Exercise 6

Solve the differential equation.

$$9y'' + 4y = 0$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad y' = re^{rx} \quad \rightarrow \quad y'' = r^2 e^{rx}$$

Plug these formulas into the ODE.

$$9(r^2 e^{rx}) + 4(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$9r^2 + 4 = 0$$

Solve for r.

$$r^2 = -\frac{4}{9}$$

$$r = \left\{ \pm i\frac{2}{3} \right\}$$

Two solutions to the ODE are $e^{-2ix/3}$ and $e^{2ix/3}$. By the principle of superposition, then,

$$\begin{aligned} y(x) &= C_1 e^{-2ix/3} + C_2 e^{2ix/3} \\ &= C_1 \left(\cos\frac{2x}{3} - i\sin\frac{2x}{3} \right) + C_2 \left(\cos\frac{2x}{3} + i\sin\frac{2x}{3} \right) \\ &= (C_1 + C_2)\cos\frac{2x}{3} + (-iC_1 + iC_2)\sin\frac{2x}{3} \\ &= C_3 \cos\frac{2x}{3} + C_4 \sin\frac{2x}{3}, \end{aligned}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.